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EXACT LIMITS

Charles Ching-an CHENG
The Institute for Advanced Study, Princeton, NJ 08540, and
Oakland University, Rochester, MI 48063, USA

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Let R be a ring with identity and let $\mathbb{C}$ be a small category. We shall denote the category of left R-modules by $\mathcal{M}$ and the category of functors $D:\mathbb{C}\to\mathcal{M}$ by $\mathcal{M}^{\mathbb{C}}$. The limit functor $\mathcal{M}^{\mathbb{C}}\to\mathcal{M}$ is always left exact. Therefore it is natural to ask when it will be exact [6]. If we denote the constant R-valued functor in $\mathcal{M}^{\mathbb{C}}$ by ΔR then it is not hard to show that the limit functor is exact if and only if ΔR is projective. Let the *R-cohomological dimension* $\mathbb{C}$, denoted by $\mathrm{cd}_R\,\mathbb{C}$, be defined as the projective dimension of ΔR. Then the above problem amounts to characterizing all small categories $\mathbb{C}$ with $\mathrm{cd}_R\,\mathbb{C}=0$. Partial solutions exist in the following cases:

1. $\mathbb{C}$ is a group.
2. $\mathbb{C}$ is a finitely generated abelian monoid.
3. $R=\mathbb{Z}$.

Let A be a $\mathbb{C}$-set (i.e. a functor $A:\mathbb{C}\to\mathcal{S}ets$). Then one can form a functor $RA:\mathbb{C}\to\mathcal{M}$ whose value at p is the free R-module on $A(p)$. If each $A(P)$ is a singleton then $RA\cong\Delta R$. Therefore one may ask a more general question: When is RA projective? In this paper we settle this in case $A(\alpha)$ is injective for all morphisms α of $\mathbb{C}$. The result extends Theorem 4.2 of [1] and gives rise to a solution of the problem on exact limits.

The Yoneda functor $Y:\mathbb{C}^{\mathrm{op}}\to\mathcal{S}ets^{\mathbb{C}}$ is defined by $Y(p)=\mathbb{C}(p,\ )$. Let A be a $\mathbb{C}$-set and let (Y,A) be the comma category. Then the objects of $(Y,A)^{\mathrm{op}}$ can be identified with the elements of A (i.e. elements of $A(p)$) and the morphisms from x to y can be identified with the morphisms α of $\mathbb{C}$ such that $\alpha x=y$ (i.e. $A(\alpha)x=y$). Note that if each $A(p)$ is a singleton then $RA\cong\Delta R$ and $(Y,A)^{\mathrm{op}}\cong\mathbb{C}$.

A $\mathbb{C}$-set is *decomposable* if it is a disjoint union of two $\mathbb{C}$-sets. It is *indecomposable* if it is not decomposable. Every $\mathbb{C}$-set A is a disjoint union of indecomposable $\mathbb{C}$-sets A_i. In this case, (Y,A) has components (Y,A_i) and RA is the direct sum of RA_i. Therefore, in determining the projectivity of RA, one may assume that A is indecomposable, i.e. (Y,A) is connected.

Theorem. *Suppose A is an indecomposable $\mathbb{C}$-set such that $A(\alpha)$ is injective for all α*

in $\mathbb{C}$. Then RA is projective if and only if there exists an object e of $(Y,A)^{\mathrm{op}}$ which maps to all objects such that $(Y,A)^{\mathrm{op}}(e,e)$ contains a finite subset E satisfying:

(a) *$\alpha E=\beta E$ for all morphisms α, β whenever the equation makes sense (i.e. α, β have the same domain e and a common codomain).*

(b) *The order of E is invertible in R.*

Proof. Consider the epimorphism

$$\pi : \bigoplus R\mathbb{C}(|x|,\ )\to RA$$

where the direct sum is indexed by elements x of A, $|x|$ denotes the object p such that $x\in A(p)$, and where the xth coordinate of π is induced by x. Then RA is projective if and only if π splits.

Suppose RA is projective and suppose μ is a splitting map for π. Then, for each element z of A,

$$\mu(z)=\sum r_{\alpha,x,z}(\alpha,x). \tag{1}$$

where $r_{\alpha,x,z}\in R$, (α,x) denotes the morphism $\alpha\in\mathbb{C}(|x|,|z|)$ in the xth component of the direct sum,

$$\sum r_{\alpha,x,z}\alpha x=z, \tag{2}$$

and if $\beta y=z$ then

$$\sum r_{\alpha,x,y}(\beta\alpha,x)=\sum r_{\alpha,x,z}(\alpha,x). \tag{3}$$

Under the assumption that $A(\alpha)$ is injective for all α we may assume that the sum in (1) contains only those terms (α,x) with $\alpha x=z$, since if $\beta\alpha x=z=\beta y$, then $\alpha x=y$.

We say that x dominates z if $\sum r_{\alpha,x,z}\neq 0$. Then (2) implies that every z is dominated by some element and (3) implies that x dominates y if and only if it dominates βy. Since A is indecomposable, it follows that any element of A that dominates an element must dominate all elements of A. Let e be such an element and consider the following epimorphism:

$$\pi' : R\mathbb{C}(|e|,\ )\to RA$$

induced by e. Since RA is projective, π' has a splitting μ'. We may assume, as before, that $\mu'(e)$ contains only those terms α with $\alpha e=e$. Let S be the set of all these α. Then S is a finite subset of $M=\{\alpha \mid \alpha e=e\}$ such that $\alpha S=\beta S$ for all morphisms α, β with the same domain e and a common codomain. In particular, $\alpha S=S$ for all $\alpha\in S$ and so S is a subsemigroup of M. Thus the hypothesis of the following lemma is satisfied.

Lemma 1 (Suschkewitch [7]). *Suppose S is a finite semigroup with $\alpha S=S$ for all α in S. Then S is isomorphic to the direct product of $S\varepsilon$ and T where ε is an idempotent of S, $S\varepsilon$ is a group with identity ε and T is the subsemigroup of all idempotents of S.*

Remark 1. For a generalization of the above lemma, the reader may consult [3, page 38]. Note, also, that the group reflection (or, the group of quotients) $\hat{S}$ of S, in this case, is isomorphic to $S\varepsilon$.

Let $E = S\varepsilon$. Then $\alpha E = \beta E$ for all morphisms α, β with the same domain e and a common codomain. It remains to show that the order of E is invertible in R. Note that the image of the composition

$$RS \xrightarrow{i} RM \xrightarrow{j} R\mathbb{C}(|e|, |e|) \xrightarrow{\pi'_{|e|}} RA(|e|),$$

where i, j denote the natural inclusions, is the trivial RS-module R (i.e. $x \circ 1 = 1$ for all x in S). Since $\mu'(e) \in RS$, it induces a splitting for the augmentation map $RS \to R$ taking $\sum r_i x_i$ to $\sum r_i$. Consider the commutative diagram

$$\begin{tikzcd} RS \arrow[r, "\Phi"] \arrow[d, "u"'] & R \arrow[d, equal] \\ R\hat{S} \arrow[r, "\hat{\Phi}"] & R \end{tikzcd}$$

where Φ are $\hat{\Phi}$ are the augmentation maps and where u is induced by the canonical map $S \to \hat{S}$. Since Φ has a splitting ψ, it is not hard to check that $u\psi$ is a splitting map for $\hat{\Phi}$. Thus $\mathrm{cd}_R \hat{S} = 0$ and so $\hat{S}$ has its order invertible in R. By Remark 1, one sees that E has its order invertible in R.

To prove the 'if' part of the theorem we define a map $\mu' : RA \to R\mathbb{C}(|e|, \ )$ by

$$\mu'(e) = \frac{1}{|E|} \sum_{\alpha \in E} \alpha.$$

Clearly μ' is a splitting map for π'.

Remark 2. It follows from the proof that the finite subset E in the theorem can be chosen to be a subsemigroup of M isomorphic to a group.

Remark 3. An example of [1] shows that the condition "$A(\alpha)$ is injective for all α" can not be removed from the theorem.

Recall that a category has a *right zero* if there exists an endomorphism ε of an object that maps to all objects such that $\alpha\varepsilon = \beta\varepsilon$ whenever the equation makes sense.

Taking $R = \mathbb{Z}$ we obtain the following result of [1] from the theorem.

Corollary 1 (Cheng–Mitchell). *Suppose A is an indecomposable $\mathbb{C}$-set such that $A(\alpha)$ is injective for all α. Then $\mathbb{Z}A$ is projective if and only if $(Y, A)^{\mathrm{op}}$ has a right zero.*

Corollary 2. *Suppose A is a $\mathbb{C}$-set with $A(\alpha)$ injective for all α. Suppose $\mathbb{C}$ contains no nontrivial isomorphisms and idempotents. If RA is projective then it is free.*

Proof. As before, we may assume that A is indecomposable and, therefore, it is enough to show that A is representable. By the theorem, there exists a finite subset $E=S\varepsilon$ of M such that $E\cong\hat{S}$, where $M=(Y,A)^{\mathrm{op}}(e,e)$. Since $\mathbb{C}$ has no nontrivial idempotents, $\varepsilon=1$ and so E is a subgroup of M. Since the only isomorphisms of $\mathbb{C}$ are the identities, $E=1$. Hence if $\alpha e=\beta e$ then $\alpha E=\beta E$, i.e. $\alpha=\beta$. Thus $(Y,A)^{\mathrm{op}}$ has an initial object e and so $A\cong\mathbb{C}(|e|,\ )$.

Taking RA to be ΔR in the theorem, one obtains the following solution to the problem on exact limits.

Corollary 3. *Let $\mathbb{C}$ be a connected small category and let R be a ring with identity. Then $\mathrm{cd}_R\,\mathbb{C}=0$ if and only if there exists an object e of $\mathbb{C}$ that maps to all objects such that $\mathbb{C}(e,e)$ contains a finite subset E satisfying:*

(a) *$\alpha E=\beta E$ whenever the equation makes sense.*

(b) *The order of E is invertible in R.*

In the remainder of this paper we shall deduce from the above corollary all the partial results on the problem of exact limits.

Corollary 4 (Laudal [5]). *The limit functor $\mathcal{A}b^{\mathbb{C}}\to\mathcal{A}b$ is exact if and only if the components of $\mathbb{C}$ have right zeros.*

Proof. Take $R=\mathbb{Z}$ in Corollary 3 and recall that the limit functor is exact if and only if $\mathrm{cd}_{\mathbb{Z}}\,\mathbb{C}=0$.

For convenience we restate Corollary 3 in case $\mathbb{C}$ has only one object (i.e. $\mathbb{C}$ is a monoid M).

Corollary 5. *Suppose M is a monoid. Then $\mathrm{cd}_R\,M=0$ if and only if there exists a finite subset E of M such that:*

(a) *$mE=E$ for all $m\in M$.*

(b) *The order of E is invertible in R.*

Now we are ready to deduce the other two results on exact limits.

Corollary 6. *Suppose G is a group. Then $\mathrm{cd}_R\,G=0$ if and only if G is finite with its order invertible in R.*

Proof. Take $M=G$ in Corollary 5. If $x\in E$ then $x^{-1}E=E$ implies that $1\in E$. This, in turn, implies that $E=G$ and so the result follows.

Remark 4. The above corollary is not hard to prove directly (e.g. see [4]). This is why we have used it in the proof of the theorem.

Corollary 7 (Cheng–Shapiro [2]). *Suppose M is a finitely generated abelian monoid. Then* $\mathrm{cd}_R M=0$ *if and only if* $\mathrm{cd}_R \hat{M}=0$. *(Here, as before, $\hat{M}$ denotes the group reflection of M.)*

Proof. Using Remark 2, one sees that the necessary and sufficient condition for $\mathrm{cd}_R M=0$ in Corollary 5 implies that of this corollary since $\hat{E}$ maps onto $\hat{M}$. It remains to show that $\mathrm{cd}_R \hat{M}=0$ implies the condition in Corollary 5.

Let M be generated by elements $x_1, x_2, \dots, x_n$ as an abelian monoid. Then $\hat{M} \cong F/X$ where F is the free abelian group generated by $x_1, x_2, \dots, x_n$ and where X is the subset (subgroup) of F consisting of elements of the form

$$\left(\prod_i x_i^{e_i}\right)\left(\prod_i x_i^{f_i}\right)^{-1}$$

where $e_i, f_i \in \mathbb{Z}^+$ and $\prod_i x_i^{e_i} = \prod_i x_i^{f_i}$ in M. Since $\mathrm{cd}_R \hat{M}=0$, $\hat{M}$ is a finite group with its order invertible in R. Thus, for each i, the image of x_i in $\hat{M}$ has finite order, say p_i. Therefore

$$x_i^{p_i} = \left(\prod_j x_j^{e_{ij}}\right)\left(\prod_j x_j^{f_{ij}}\right)^{-1}$$

in F where

$$\prod_j x_j^{e_{ij}} = \prod_j x_j^{f_{ij}}$$

in M, and where $e_{ij} \in \mathbb{Z}^+$. Let

$$E_i = \{x_i^{e_{ii}}, x_i^{e_{ii}+1}, \dots, x_i^{e_{ii}+p_i-1}\} \prod_{j \neq i} x_i^{e_{ij}}.$$

Then $x_i E_i = E_i$. Set $E = E_1 E_2 \cdots E_n$. Then, since M is abelian, $x_i E = E$ for all i and so $mE = E$ for all $m \in E$. The order of E is $\prod_i p_i$ and, therefore, is invertible in R since each p_i is.

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